

SPECTRAL ESTIMATIONS FOR LAPLACE OPERATOR FOR THE DISCRETE HEISENBERG GROUP

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We estimate the spectral measure of Laplace operator $\Delta = \frac{1}{4}(x + x^{-1} + y + y^{-1})$ for the discrete Heisenberg group with generators x and y in vicinity of the unity.

Let H be the discrete Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & k & m \\ 0 & 1 & \ell \\ 0 & 0 & 1 \end{pmatrix}, \quad k, \ell, m \in \mathbb{Z} \right\}.$$

We will call analogous matrix group with elements from $\mathbb{Z}/n\mathbb{Z}$ a finite Heisenberg group and denote it H_n . The generators of H and H_n will be denoted by the same letters:

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (1)$$

The element Δ of the group algebra for H , $\Delta = \frac{1}{4}(x + x^{-1} + y + y^{-1})$, is the Laplace operator corresponding to the system of generators (x, y) . Analogously, $\Delta_n = \frac{1}{4}(x + x^{-1} + y + y^{-1})$ is the Laplace operator for the finite Heisenberg group H_n . These operators can also be defined as transition operators for random walk on the groups. We will deal with spectra of these operators in the regular representations of the corresponding groups.

It is easy to demonstrate [BVZ] that the spectrum of Δ is the interval $[-1, 1]$. Let E_A , $A \subset [-1, 1]$ be a family of spectral projectors for Δ and $\mu A = (E_A \delta_e, \delta_e)$ be the corresponding spectral measure. Here $\delta_e \in L_2(H)$ is the characteristic function of the unit element of the group H . We will estimate the value $\mu([-1, -1+t] \cup [1-t, 1])$ for $t \rightarrow 0$. More precisely we will prove the inequality $\mu([-1, -1+t] \cup [1-t, 1]) \geq \text{const } t^{2+\alpha}$.

The paper is organized as follows. The first section contains description of the representations of the finite Heisenberg group. In the second section we demonstrate that all eigenvalues of the operator Δ_n in multidimensional representations are less than $1 - O(\frac{1}{n})$. In the third section we give a combinatorial realization for the characteristic polynomials of Δ_n in the multidimensional representations. And in the fourth section we prove the estimates for the spectral measure of Δ , using those for the finite group.

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1. Representations of the finite Heisenberg group H_n .

We will consider only the simplest case: n is a prime number. In this case the group H_n has n^2 one-dimensional representations $T_{\alpha, \beta}$, $\alpha, \beta = 1, 2, \dots, n$:

$$T_{\alpha, \beta}(x) = e^{\frac{2\pi i}{n}\alpha}, \quad T_{\alpha, \beta}(y) = e^{\frac{2\pi i}{n}\beta}, \quad T_{\alpha, \beta}(z) = 1.$$

Besides there are $n - 1$ irreducible representations T_q of dimension n . These representations can be obtained by induction from the representations ρ_q of the abelian subgroup generated by the elements y and z , where $\rho_q(y) = 1$, $\rho_q(z) = e^{\frac{2\pi i}{n}q}$, $q = 1, \dots, n - 1$. The representation T_q can be described in the space \mathbb{C}^n with the basis u_1, u_2, \dots, u_n by the formulae

$$T_q(x)u_j = u_{j+1}, \quad T_q(y)u_j = e^{\frac{2\pi i}{n}qj}u_j, \quad T_q(z)u_j = e^{\frac{2\pi i}{n}q}u_j.$$

Proposition 1.1. $T_{\alpha,\beta}$ and T_q is a complete set of irreducible unitary representations of the group H_n .

Indeed, irreducibility and nonequivalence of the representations T_q can be verified by computing scalar products of the corresponding characters. Computing squares of the dimensions of the representations $T_{\alpha,\beta}$ and T_q we can see that the set is complete.

Let $\Delta_n = \frac{1}{4}(x + y + x^{-1} + y^{-1}) \in \mathbb{C}H_n$ be the Laplace operator corresponding to the generators (1). Denote $\tilde{\Delta}_n = 4\Delta_n$. In order to find the spectrum of $\tilde{\Delta}_n$ in the regular representation it suffices to find its spectra in all irreducible representations. In the case of the one-dimensional representations $T_{\alpha,\beta}$ the operator is

$$T_{\alpha,\beta}(\tilde{\Delta}_n) = 2 \cos \frac{2\pi\alpha}{n} + 2 \cos \frac{2\pi\beta}{n}.$$

For multidimensional representations T_q the operator has a more complicated form

$$T_q(\tilde{\Delta}_n) = \begin{pmatrix} 2 \cos \frac{0 \cdot 2q\pi}{n} & 1 & 0 & 0 & \dots & 1 \\ 1 & 2 \cos \frac{1 \cdot 2q\pi}{n} & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 \cos \frac{2 \cdot 2q\pi}{n} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 2 \cos \frac{(n-2) \cdot 2q\pi}{n} & 1 \\ 1 & 0 & \dots & 0 & 1 & 2 \cos \frac{(n-1) \cdot 2q\pi}{n} \end{pmatrix}$$

Unfortunately the spectra of these matrices cannot be computed directly.

Such matrices appear in investigations of the Harper operator and the Mathieu equation (see e.g. [AMS, BVZ]). Consider the Harper operator in the space $L^2(\mathbb{Z})$

$$u_n \mapsto u_{n+1} + u_{n-1} + 2 \cos(2\pi n\alpha)u_n.$$

Specialists in the high energy physics know very well the so called Hofstadter butterfly [H], depicted in the figure 1 (we have taken this picture from [IMOS]). Here the horizontal axis corresponds to the parameter α , and the vertical axis — to the spectrum. If $\alpha = \frac{q}{n}$ is a rational number (irreducible fraction), then the spectrum consists of n intervals of the form $f^{-1}([-4, 0])$, where f is the characteristic polynomial of the matrix $T_q(\Delta_n)$ (if n is even the $\frac{n}{2}$ -th and $(\frac{n}{2} + 1)$ -th intervals have the common vertex 0). If α is irrational, the spectrum is a Cantor set.

The endpoints of the intervals form curves well seen in the picture. The curve marked by the arrow has a slope $c \neq 0$ and it implies that $\rho_n \sim 4 - \frac{c}{n}$, where ρ_n is the spectral radius of the Harper operator for $\alpha = \frac{1}{n}$. In fact $\rho_n = \lambda_n$, where λ_n is the maximal eigenvalue of the matrix $T_1(\Delta_n)$. In the next section we will show that $4 - \lambda_n = O(\frac{1}{n})$, which agrees with the form of the marked curve.

Numerical computations allow us to conjecture that $\lambda_n = 4 - \frac{2\pi}{n} + o(\frac{1}{n})$ when $n \rightarrow +\infty$. For instance, for $n = 1000$ the value of $(4 - \lambda_n)n$ is approximately 6.28 (the next digit does not fit 2π ; this has been computed by means of the Mathematica 3.0).

2. Estimation of the maximal eigenvalue of the operator $\tilde{\Delta}_n$ in the representation T_q .

Notations. Matrices are denoted by “calligraphic” letters — \mathcal{A} , \mathcal{B} etc. The maximal eigenvalue of \mathcal{A} is denoted $\lambda_{\mathcal{A}}$. Let $\mathcal{P} = (p_{ij})$ and $\mathcal{Q} = (q_{ij})$ be real $n \times n$ matrices. We write $\mathcal{P} \leq \mathcal{Q}$, if $p_{ij} \leq q_{ij}$ for all i, j .

Let \mathcal{A}_n be $n \times n$ matrix defining our operator $\tilde{\Delta}_n$ in the representation T_1 :

$$\mathcal{A}_n = \begin{pmatrix} 2 \cos \frac{0 \cdot 2\pi}{n} & 1 & 0 & 0 & \dots & 1 \\ 1 & 2 \cos \frac{1 \cdot 2\pi}{n} & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 \cos \frac{2 \cdot 2\pi}{n} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 2 \cos \frac{(n-2) \cdot 2\pi}{n} & 1 \\ 1 & 0 & \dots & 0 & 1 & 2 \cos \frac{(n-1) \cdot 2\pi}{n} \end{pmatrix}$$

This is a symmetric matrix, it has a real spectrum. Let λ_n be its maximal eigenvalue. It is convenient to deal with the matrix $\tilde{\mathcal{A}}_n = 2\mathcal{E}_n + \mathcal{A}_n$ (\mathcal{E}_n is the $n \times n$ identity matrix), with non-negative entries, and the spectrum is shifted by 2 with respect to the spectrum of \mathcal{A}_n , and the set of eigenvectors coincides with that of \mathcal{A}_n . In particular, $\tilde{\lambda}_n = \lambda_n + 2$ where $\tilde{\lambda}_n$ is the maximal eigenvalue of $\tilde{\mathcal{A}}_n$. According to the Perron-Frobenius theorem the maximal eigenvalue of an irredundant non-negative matrix has the multiplicity 1, hence λ_n is an eigenvalue of multiplicity 1.

We need also the following consequence of the Perron-Frobenius theorem:

Proposition 2.1. *Let \mathcal{P} and \mathcal{Q} be $n \times n$ matrices with non-negative entries, $\mathcal{P} \leq \mathcal{Q}$, then $\lambda_{\mathcal{P}} \leq \lambda_{\mathcal{Q}}$.*

Let us demonstrate that $4 - \lambda_n = O(\frac{1}{n})$ when $n \rightarrow +\infty$. We choose the numbers $\sqrt{n/2}$, $37/15$ etc in the proofs of the next two lemmas in such a way that we obtain good approximations of exact constants.

Lemma 2.2. $4 - \frac{40}{n} < \lambda_n < 4 - \frac{2}{n}$.

Proof. 1) The left inequality.

Since it does not affect the final result, we will treat the expression $\sqrt{n/2}$ as even integer. Let $\mathcal{C}_{\sqrt{n/2}}$ be tridiagonal $\sqrt{n/2} \times \sqrt{n/2}$ matrix of the form

$$\mathcal{C}_{\sqrt{n/2}} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

It is well known (see e.g. [GHJ]) that the spectrum of $\mathcal{C}_{\sqrt{n/2}}$ is the set $\{2 \cos \frac{j\pi}{\sqrt{n/2}+1}, 1 \leq j \leq \sqrt{n/2}\}$. Let \mathcal{B}_n be $n \times n$ matrix having the following block form:

$$\mathcal{B}_n = \begin{pmatrix} (4 - \frac{20}{n})\mathcal{E}_{\sqrt{n/2}} + \mathcal{C}_{\sqrt{n/2}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (4 - \frac{20}{n})\mathcal{E}_{\sqrt{n/2}} + \mathcal{C}_{\sqrt{n/2}} \end{pmatrix}.$$

Then $\tilde{\mathcal{A}}_n \geq \mathcal{B}_n$ (since $2 \cos \frac{2\pi k}{n} \geq 2 - \frac{2\pi^2}{n} > 2 - \frac{20}{n}$ for $|k| \leq \sqrt{n/2}$). By the proposition 2.1 we get

$$\lambda_n \geq \lambda_{\mathcal{B}_n} - 2 = 2 - \frac{20}{n} + 2 \cos \frac{\pi}{\sqrt{n/2}+1} > 4 - \frac{40}{n}.$$

2) The right inequality. To simplify notations we will treat the expression $\sqrt{n}/4$ as integer.

Let \mathcal{D}_n be “generalized”-tridiagonal matrix, containing on the main diagonal the numbers $\frac{37}{15n}$ ($\sqrt{n}/4$ times), then $n+1 - 2\sqrt{n}/4$ zeros, and then again the numbers $\frac{37}{15n}$ ($\sqrt{n}/4 - 1$ times):

$$\mathcal{D}_n = \begin{pmatrix} \frac{37}{15n} & 1 & & & & & & & & & 1 \\ 1 & \frac{37}{15n} & 1 & & & & & & & & \\ & \dots & \dots & \dots & & & & & & & \\ & & 1 & \frac{37}{15n} & 1 & & & & & & \\ & & & 1 & 0 & 1 & & & & & \\ & & & & 1 & 0 & 1 & & & & \\ & & & & & \dots & \dots & \dots & & & \\ & & & & & & 1 & 0 & 1 & & \\ & & & & & & & 1 & \frac{37}{15n} & 1 & \\ & & & & & & & & \dots & \dots & \\ & & & & & & & & & \frac{37}{15n} & \\ 1 & & & & & & & & & 1 & \frac{37}{15n} \end{pmatrix}$$

Then the inequality

$$(2) \quad \tilde{\mathcal{A}}_n \leq (4 - \frac{37}{15n})\mathcal{E}_n + \mathcal{D}_n,$$

holds, because for large n and $\sqrt{n}/4 \leq k \leq n - \sqrt{n}/4$ the estimation $2 \cos \frac{2\pi k}{n} \leq 2 - \frac{\pi^2}{4n} + O(\frac{1}{n^4}) < 2 - \frac{37}{15n}$ is valid since $\pi^2/4 \approx 2.467 > 37/15 = 2.466$.

Let us find an upper bound for the maximal eigenvalue $\lambda_{\mathcal{D}_n}$. It is clear, that, $2 < \lambda_{\mathcal{D}_n} < 2 + \frac{37}{15n}$. Let $\lambda_{\mathcal{D}_n} = 2 + \alpha$, where $0 < \alpha < \frac{37}{15n}$. We will obtain below more precise estimation $\alpha < \frac{5}{11n} = \frac{0.454}{n}$. Then the statement of the lemma will be a direct consequence of the inequality (2).

We will give a proof from the contrary. Assume that $\alpha > \frac{5}{11n}$ (the only place where we use this assumption is the 4th step of the proof). Using this assumption in technical calculations, we will obtain the inequality $\alpha < \frac{5}{11n}$. This contradiction implies that actually $\alpha < \frac{5}{11n}$.

Let $\mathbf{v} = (v_1, v_2, \dots, v_n)$ be the eigenvector corresponding to the eigenvalue $\lambda_{\mathcal{D}_n}$:

$$(3) \quad \mathcal{D}_n \mathbf{v} = (2 + \alpha) \mathbf{v}.$$

We define v_k for all $k \in \mathbb{Z}$ by the periodicity. We normalize the vector \mathbf{v} by the condition $\max_k v_k = 1$. Recall that all the v_i are non-negative.

Let us list some properties of \mathbf{v} .

1. \mathbf{v} is an eigenvector of multiplicity 1. Therefore it is symmetric: $v_k = v_{n+2-k}$ (here $v_{n+1} = v_1$).
2. $\max_k v_k = v_1 = 1$, $\min_k v_k = v_{n/2}$.

Indeed, for $-\sqrt{n}/4 + 1 \leq k \leq \sqrt{n}/4$ it follows from relation (3) that

$$(4) \quad v_{k-1} + v_{k+1} = \left(2 - \frac{37}{15n} + \alpha\right) v_k,$$

in other words $v_{k-1} + v_{k+1} < 2v_k$, and v_k cannot be a local minimum. Analogously, for $\sqrt{n}/4 < k \leq n - \sqrt{n}/4$

$$(5) \quad v_{k-1} + v_{k+1} = (2 + \alpha) v_k,$$

hence $v_{k-1} + v_{k+1} > 2v_k$, and v_k cannot be a local maximum. Now the statement follows from the symmetry of \mathbf{v} .

3. $v_{\sqrt{n}/4} = v_{n-\sqrt{n}/4+2} > \frac{12}{13}$ for sufficiently large n .

Indeed, from the equality (4) we get

$$v_k - v_{k+1} = \left(\frac{37}{15n} - \alpha\right) v_k + (v_{k-1} - v_k) \leq \frac{37}{15n} + (v_{k-1} - v_k).$$

Summing up these relations, we obtain

$$v_k - v_{k+1} \leq \frac{37}{15n}(k-1) + (v_1 - v_2),$$

therefore

$$v_1 - v_{\sqrt{n}/4} \leq \frac{37}{15n} \left(1 + 2 + \dots + (\sqrt{n}/4 - 1)\right) + (\sqrt{n}/4 - 1)(v_1 - v_2).$$

Finally, notice that the first term does not exceed $\frac{1}{32} \cdot \frac{37}{15} = \frac{37}{480} < \frac{1}{13}$, and the second term is less than $\frac{37}{120\sqrt{n}}$ (it follows from (4) for $k = 1$ and from the symmetry of \mathbf{v}). Thus, $v_{\sqrt{n}/4} > \frac{12}{13}$ for large n .

4. For large n $v_{\sqrt{n}/4+1} + v_{\sqrt{n}/4+2} + \dots + v_{n-\sqrt{n}/4} > \frac{24}{13\sqrt{n}}$.

Indeed, solving homogeneous difference equation (5) and taking into account the symmetry of \mathbf{v} , we deduce the formulae

$$v_{n/2+k} = C_0 \left(\left(1 + \alpha/2 + \sqrt{\alpha + \alpha^2/4}\right)^k + \left(1 + \alpha/2 - \sqrt{\alpha + \alpha^2/4}\right)^k \right) \quad (-n/2 + \sqrt{n}/4 \leq k \leq n/2 - \sqrt{n}/4).$$

So, we need only to estimate the sum of geometric progression. Denote for brevity $N = n/2 - \sqrt{n}/4$, $x = 1 + \alpha/2 + \sqrt{\alpha + \alpha^2/4}$, then $x^{-1} = 1 + \alpha/2 - \sqrt{\alpha + \alpha^2/4}$,

$$(6) \quad v_{\sqrt{n}/4+1} + v_{\sqrt{n}/4+2} + \dots + v_{n-\sqrt{n}/4} = C_0 \sum_{k=-N}^N (x^k + x^{-k}) = 2C_0 \frac{x^{N+1/2} - x^{-(N+1/2)}}{x^{1/2} - x^{-1/2}}.$$

By the assumption, $\frac{5}{11n} < \alpha < \frac{37}{15n}$, $x^N = e^{\text{const}\sqrt{n}}$, $x^{-N} = e^{-\text{const}\sqrt{n}}$, therefore $x^{N+1/2} - x^{-(N+1/2)} \geq (1 - \varepsilon)x^{N+1/2}$ for any positive ε , if n is sufficiently large. Notice also that

$$\frac{1}{x^{1/2} - x^{-1/2}} = \frac{x^{1/2} + x^{-1/2}}{x - x^{-1}} \geq \frac{1}{\sqrt{\alpha + \alpha^2/4}} \geq \frac{1 - \varepsilon}{\sqrt{\alpha}}.$$

Now we can obtain a lower bound for the r.h.s. in the equality (6):

$$\sum_{k=-N}^N v_{n/2+k} = 2C_0 \frac{x^{N+1/2} - x^{-(N+1/2)}}{x^{1/2} - x^{-1/2}} \geq (1 - \varepsilon)^2 \frac{2C_0 x^N x^{1/2}}{\sqrt{\alpha}} = (1 - \varepsilon)^2 \frac{2v\sqrt{n}/4}{\sqrt{\alpha}} > \frac{24}{13\sqrt{\alpha}},$$

(ε should be chosen sufficiently small in order to the last inequality was correct according to the 3rd step of the proof).

To complete the proof of lemma 2, sum up equalities (4) and (5) for all k :

$$\frac{37}{15n} \sum_{k=-\sqrt{n}/4+1}^{\sqrt{n}/4} v_k = \alpha \sum_{k=1}^n v_k$$

Due to normalization $0 \leq v_i \leq 1$ and the inequality given in the 4th step, we easily estimate both sides

$$2 \cdot \frac{37}{15n} \sqrt{n}/4 > \frac{37}{15n} \sum_{k=-\sqrt{n}/4+1}^{\sqrt{n}/4} v_k \geq \alpha \sum_{k=\sqrt{n}/4+1}^{n-\sqrt{n}/4} v_k > \frac{24}{13} \sqrt{\alpha}.$$

Thus,

$$\alpha \leq \left(\frac{37 \cdot 2 \cdot 13}{24 \cdot 15 \cdot 4} \right)^2 \cdot \frac{1}{n} < \frac{4}{11n}.$$

□ Lemma 2.2 is proven.

For the maximal eigenvalues of the operator $\tilde{\Delta}_n$ in the other representations T_k we will give only an upper bound. It is worse than that of lemma 2.2, however we believe that λ_n in lemma 2.2 is the maximal eigenvalue among the eigenvalues of matrices $T_q(\Delta_n)$ for all q . Let μ_n be the maximal eigenvalue of the operator $\tilde{\Delta}_n$ in any of representations T_q .

Lemma 2.3. $\mu_n \leq 4 - \frac{3}{5n}$.

Proof. Similarly to the proof of the lemma 2.2, we choose a matrix which majorizes $T_q(\tilde{\Delta}_n)$. For this purpose we replace $\sqrt{n}/3$ largest expressions of the form $2 \cos \frac{2\pi qj}{n}$ at the diagonal, with 2 and all the other diagonal entries with $2 - 1/n$. Then we have the following inequality $T_q(\tilde{\Delta}_n) + 2\mathcal{E}_n \leq (4 - \frac{1}{n})\mathcal{E}_n + \tilde{\mathcal{D}}_n$, since $\cos \frac{2\pi\sqrt{n}/6}{n} \leq 2 - \frac{\pi^2}{9n} + O(\frac{1}{n^4}) < 2 - \frac{1}{n}$ for large n . Here $\tilde{\mathcal{D}}_n$ is a matrix analogous to \mathcal{D}_n , having $\sqrt{n}/3$ numbers $1/n$ at the diagonal in the places corresponding to large cosines, and zeros in the other places. Let $\mathbf{v} = (v_1, v_2, \dots, v_n)$ be the eigenvector for the number $\lambda_{\tilde{\mathcal{D}}_n}$, normalized by condition $\max_k v_k = 1$. As above, denote $\lambda_{\tilde{\mathcal{D}}_n} = 2 + \alpha$. To prove the lemma it suffices to verify that $\alpha \leq \frac{2}{5n}$.

Components of vector \mathbf{v} obey the relations

$$v_{k-1} + v_{k+1} = \left(2 - \frac{1}{n} + \alpha\right)v_k \quad \text{or} \quad v_{k-1} + v_{k+1} = (2 + \alpha)v_k.$$

Multiply each relation by v_k and sum over all k :

$$2 \sum_{k=1}^n v_k v_{k+1} = -\frac{1}{n} \sum' v_k^2 + (2 + \alpha) \sum_{k=1}^n v_k^2,$$

where \sum' denotes the summation over those k , for which the k -th row of the matrix $\tilde{\mathcal{D}}_n$ contains $\frac{1}{n}$ in the diagonal (recall that there are $\sqrt{n}/3$ such k). Rewrite the last equality in the form

$$(7) \quad \sum_{k=1}^n (v_k - v_{k+1})^2 + \alpha \sum_{k=1}^n v_k^2 = \frac{1}{n} \sum' v_k^2.$$

An upper bound for the r.h.s. is evident:

$$(8) \quad \frac{1}{n} \sum' v_k^2 \leq \frac{1}{n} \cdot \frac{\sqrt{n}}{3} = \frac{1}{3\sqrt{n}}.$$

Using, if necessary, the cyclic permutation of the coordinates, we can think that $v_1 = \max_k v_k$. Then for any s we have the following lower bound for the l.h.s.:

$$(9) \quad \sum_{k=1}^n (v_k - v_{k+1})^2 + \alpha \sum_{k=1}^n v_k^2 \geq \sum_{k=1}^s (v_k - v_{k+1})^2 \geq \frac{(1 - v_s)^2}{s}.$$

The second inequality here is the inequality about the arithmetic mean and the quadratic mean (or its simple consequence, if some of the differences $v_k - v_{k+1}$ are negative). Combining (7), (8) and (9), we obtain inequality

$$(1 - v_s)^2 \leq \frac{s}{3\sqrt{n}},$$

hence, taking into account that $0 \leq v_s \leq 1$,

$$v_s \geq 1 - \sqrt{\frac{s}{3\sqrt{n}}}.$$

As in the proof of the previous lemma we define v_k for all $k \in \mathbb{Z}$ by the periodicity. Then for $|s| \leq \frac{5}{4}\sqrt{n}$

$$v_s \geq 1 - \sqrt{\frac{|s|}{3\sqrt{n}}} \geq 1 - \sqrt{5/12}.$$

Now we can give one more lower bound for the equality (7):

$$\frac{1}{3\sqrt{n}} \geq \frac{1}{n} \sum' v_k^2 = \sum_{k=1}^n (v_k - v_{k+1})^2 + \alpha \sum_{k=1}^n v_k^2 \geq \alpha \sum_{k=-\frac{5}{4}\sqrt{n}}^{\frac{5}{4}\sqrt{n}} v_k^2 \geq \left(1 - \sqrt{5/12}\right) \alpha \cdot \frac{5}{2}\sqrt{n}.$$

Thus,

$$\alpha \leq \frac{2}{15(1 - \sqrt{5/12})} \cdot \frac{1}{n} \approx 0.376 \cdot \frac{1}{n} < \frac{2}{5n}.$$

□ Lemma 2.3 is proven.

3. Combinatorial realization of the characteristic polynomial.

In this section we give a combinatorial realization of the characteristic polynomial $P_{n,q}$ of the matrices $T_q(\tilde{\Delta}_n)$.

Consider graph which is a cycle with n vertices. Enumerate its vertices along the cycle, assign the weight $x - 2 \cos \frac{2\pi qk}{n}$ to the k -th vertex and weight -1 to each edge. We denote this labeled graph $\Gamma_{n,q}$. Assign to an arbitrary set ξ of several non-intersecting edges of $\Gamma_{n,q}$ the weight w_ξ , defined as a product of weights of all edges, belonging to this set, and weights of all vertices not belonging to these edges. For example, a graph $\Gamma_{5,1}$ is drawn on the figure 2. The weight of the set, consisting of two bold edges equals to $x - 2 \cos \frac{8\pi}{5}$, and the weight of the empty set is $\prod_{k=1}^5 (x - 2 \cos \frac{2\pi k}{5})$. Denote $K_{n,q}(x) = \sum_\xi w_\xi$, where the summation runs over all possible sets of non-intersecting edges.

Lemma 3.1. $P_{n,q}(x) = K_{n,q}(x) + 2(-1)^n$.

Proof. Denote for shortness $x\mathcal{E}_n - T_q(\Delta_n) = (a_{ij})$. Then by the definition of determinant

$$(10) \quad P_{n,q} = \sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}.$$

Recall that the matrix (a_{ij}) is tridiagonal. Consider an arbitrary nonzero monomial from the r.h.s. of this equality. Assume that it contains diagonal factor a_{kk} and off-diagonal factor $a_{k+1,k+2}$. Then it necessary contains the off-diagonal factor $a_{k+2,k+1}$. It is easy to see that all off-diagonal entries of matrix (a_{ij}) enter every nonzero monomial in pairs of the form $a_{k,k+1}, a_{k+1,k}$ (excluding two monomials: $a_{12}a_{23} \dots a_{n1}$ and $a_{21}a_{32} \dots a_{1n}$). This observation allows us to construct a bijection between the sets of non-intersecting edges of our cycle $\Gamma_{n,q}$ and the nonzero monomials in the equality (10). Namely, the set $(k_1, k_1 + 1), (k_2, k_2 + 1), \dots$ corresponds to the monomial $a_{11} \dots a_{k_1-1,k_1-1} a_{k_1,k_1+1} a_{k_1+1,k_1} a_{k_1+2,k_1+2} \dots a_{k_2,k_2+1} a_{k_2+1,k_2} \dots$. The sign of this monomial is determined by the parity of number of edges and equals to the corresponding weight. The summand $2(-1)^n$ in equality (10) corresponds to the two exclusive monomials. □ Lemma 3.1 is proven.

4. An estimation of spectral measure in the case of the infinite Heisenberg group.

Let δ_e be the characteristic function of the group unity, viewed as an element of the space L^2 over this group. We will use this notation for different groups since it does not lead to a confusion. In this section we estimate the value $(E_t \delta_e, \delta_e)$, where E_t is a spectral projector of the operator Δ , corresponding to the set $[-1, -1+t] \cup (1-t, 1]$.

Remark that for the Laplace operator in \mathbb{Z}^n built from the standard generators it is easy to demonstrate with the help of Fourier transform that $(E_t \delta_e, \delta_e) \sim \text{const } t^{n/2}$.

The next simple lemma is a key element for a transition from the infinite group to a finite one.

Lemma 4.1. *Let n be arbitrary positive integer, $N = n^2 + 1$. Then $(\Delta^k \delta_e, \delta_e) = (\Delta_N^k \delta_e, \delta_e)$ for all $k \leq n$; the l.h.s. here is a scalar product in the space $L^2(H)$, and the r.h.s. — in $L^2(H_N)$.*

Proof. The operator Δ^k regarded as an element of the group algebra of the group H is a formal linear combination of matrices of the form $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$, where each matrix is a product of k generators or their inverse. The scalar product, we are interested in, equals to the coefficient of the identity matrix in such a decomposition. It is clear that the entries a and b in every such a matrix do not exceed k , and the entry c does not exceed certain quadratic function of k . It is easy to prove by induction that $|c| \leq k^2/2$. The difference between calculations for $L^2(H)$ and $L^2(H_N)$ is that the entries of matrices in H_N are not numbers but residues in the range from $-N/2$ to $N/2$. Therefore for $k \leq n$ the results for H and H_N coincide. \square Lemma 4.1 is proven.

Lemma 4.2. *For any $\alpha > 0$ there exist a constant $c_0 > 0$, arbitrary large positive integer n , and a polynomial $P_n(x)$ of degree n , such that have the following properties:*

- 1) $|P_n(x)| \leq 1$ for $|x| \leq 1$, $P_n(1) = 1$, $P_n(-1) = 1$;
- 2) $|P_n(x)| \leq \frac{1}{n^\alpha}$ for $|x| \leq 1 - \frac{1}{n^{2-\alpha}}$;
- 3) $P_n(x) \geq 0$ for $1 - \frac{1}{n^2} \leq |x| \leq 1$.

Proof. We chose P_n as a properly normalized Chebyshev polynomial. Namely, $P_n(x) = n T_n(\frac{n^{2-\alpha}}{n^{2-\alpha}-1} x)$, where C_n is a normalizing constant. For $|x| \geq 1$ the Chebyshev polynomials are given by the formulae

$$T_n(x) = \frac{1}{2} \left((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right).$$

Hence for $x = \pm 1$ and $n \rightarrow +\infty$ we derive:

$$(11) \quad P_n(x) \sim \frac{C_n}{2} \left(1 + \frac{\sqrt{2}}{n^{1-\alpha/2}} \right)^n \sim C_n e^{\sqrt{2} n^{\alpha/2}}.$$

Thus, we can take $C_n \sim 2e^{-\sqrt{2} n^{\alpha/2}}$. This guarantees not a polynomial but an exponential decreasing of $P_n(x)$ for $|x| < 1 - \frac{1}{n^{2-\alpha}}$ (more precisely for $|x| < 1 - \frac{2}{n^{2-\alpha}}$) and the separation from zero for $1 - \frac{1}{n^2} \leq |x| \leq 1$.

\square Lemma 4.2 is proven.

Remark. From minimax properties of the Chebyshev polynomials it follows that the exponent $2 - \alpha$ in the condition 2 of lemma 4.2 can not be replaced with 2. Indeed, in this case an estimation similar to (11) shows that C_n does not tend to 0, and therefore P_n , as well as any other polynomial satisfying the condition 1, is not small inside the interval $[-1, 1]$, even if the additional condition 3 is not imposed.

Theorem 4.3. *For any $\alpha > 0$ and sufficiently small t the estimation $c_1 t^{2+\alpha} \leq (E_t \delta_e, \delta_e)$ holds.*

Proof. Denote $t = 1/n^2$, $N = n^2 + 1$, let χ_t be the characteristic function for the set $A_t = [-1, -1+t] \cup [1-t, 1]$. The key idea of the proof is to substitute the function χ_t by a polynomial and then “move” all computations from the infinite Heisenberg group to the finite one.

The polynomials P_n from lemma 4.2 obey the inequalities

$$0 \chi_t(x) \leq P_n(x) \leq \chi_{t^{1-\alpha}}(x) + \frac{1}{n^6}.$$

Consequently

$$0 \mu(A_t) \leq \|P_n(\Delta)\|^2 \leq \mu(A_{t^{1-\alpha}}) + \frac{1}{n^6}.$$

Moreover, the same inequality holds for the operator Δ_N . Since $\deg P_n = n$, then in the spirit of lemma 4.1 we have

$$\|P_n(\Delta)\|^2 = \|P_n(\Delta_N)\|^2.$$

Thus,

$$(12) \quad \mu(A_{t^{1-\alpha}}) + \frac{1}{n^6} \geq \|P_n(\Delta)\|^2 = \|P_n(\Delta_N)\|^2 \geq {}_0\mu_N(A_t).$$

The r.h.s. of this inequality is evaluated in the regular representation of the finite Heisenberg group H_N . Notice, that thanks to lemma 2.3 in order to compute the r.h.s. it is sufficient to employ the information only about the one-dimensional representations of the group H_N (recall that $t = \frac{1}{n^2} \asymp \frac{1}{N}$)! Thus,

$$\mu_N(A_t) \geq \frac{1}{N^3} \sum_{s,t: \frac{s^2+t^2}{N^2} \leq \frac{1}{n^2}} 1 \geq \frac{1}{n^4}.$$

In the last inequality we estimate the number of terms as $N^2/n^2 \asymp n^2$. Together with (12) it leads to the required inequality. Formally, in order to be able to apply lemma 2.3 we must take N to be a prime number. Therefore we had to choose N more accurately, e.g., N could be chosen as a prime number in the interval $[n^2 + 1, 2n^2 + 1]$.

□ Theorem 4.3 is proven.

Unfortunately one cannot obtain the estimation $(E_t \delta_e, \delta_e) \leq c_2 t^{2-\alpha}$ in the same manner. In order to avoid multi-dimensional representations, we need to take too small t . But according to the remark after lemma 4.2, the polynomial P_n in this case does not approximate the function χ_t .

In the paper [K] the author proves the following inequality for the Laplace operator on an amenable group:

$$(E_t \delta_e, \delta_e) \geq \frac{1 - \frac{2\varepsilon}{t^2}}{|A_\varepsilon|},$$

where $\varepsilon > 0$ is any number, $|A_\varepsilon|$ is the size of Folner's ε -set. The best value of the r.h.s. for fixed t is achieved for $\varepsilon = O(t^2)$. In this case the numerator is a certain constant and the denominator is presumably $O(1/\varepsilon^4)$. This yields the estimation $(E_t \delta_e, \delta_e) \geq Ct^8$, which is worse than our result.

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